Permutations and shifts

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Abstract. The entropy of a symbolic dynamical system is usually defined in terms of the growth rate of the number of distinct allowed factors of length \( n \). Bandt, Keller and Pompe showed that, for piecewise monotone interval maps, the entropy is also given by the number of permutations defined by consecutive elements in the trajectory of a point. This result is the starting point of several works of Elizalde where he investigates permutations in shift systems, notably in full shifts and in beta-shifts. The goal of this talk is to survey Elizalde’s results. I will end by mentioning the case of negative beta-shifts, which has been simultaneously studied by Elizalde and Moore on the one hand, and by Steiner and myself on the other hand.

Keywords: Dynamical systems, permutation entropy, beta-shifts.

1 Introduction

The following result motivates the subject.

Theorem 1 (Bandt-Keller-Pompe [5]). For piecewise monotonic maps, the topological entropy coincides with the permutation entropy.

Let us introduce the permutation entropy of a totally ordered dynamical system. This notion was first introduced in [6] and then, studied in [5], [12], [13], [2] (and other papers). Let us also mention the book [1].

From now on, we suppose that \( X \) is a totally ordered set and \( T : X \to X \). For an integer \( n \geq 1 \) and a point \( x \in X \) such that \( x, T(x), \ldots, T^{n-1}(x) \) are pairwise distinct, \( \text{Pat}(T, n, x) \) denotes the permutation \( \pi \in S_n \) defined by

\[
T^{\pi^{-1}(1)-1}(x) < T^{\pi^{-1}(2)-1}(x) < \cdots < T^{\pi^{-1}(n)-1}(x).
\]

Otherwise stated, the relative order of \( x, T(x), \ldots, T^{n-1}(x) \) corresponds to the permutation \( \pi \).

Example 2. Suppose \( T^3(x) < T(x) < T^2(x) \). Then \( \text{Pat}(T, 4, x) = 3241 \).

A permutation \( \pi \) in \( S_n \) is realized, or allowed, in \((X, T)\) if there exists \( x \in X \) such that \( \text{Pat}(T, n, x) = \pi \). The set of allowed permutations of length \( n \) and the set of all allowed permutations are denoted by

\[
A(T, n) = \{ \pi \in S_n : \exists x \in X \text{ Pat}(T, n, x) = \pi \} \quad \text{and} \quad A(T) = \bigcup_{n \geq 1} A(T, n)
\]
respectively. Then the permutation entropy of \((X, T)\) is defined as
\[
\lim_{n \to \infty} \frac{1}{n} \log \# A(T, n)
\]
provided that this limit exists. Theorem 1 states that this limit exists for piecewise monotonic maps, and coincides with the topological entropy. In particular this result implies that not all permutations are realized in a given piecewise monotonic map system. In fact, most of them are not since the number of permutations of length \(n\) is super-exponential.

**Example 3 (Tent map).** Let \(X = [0, 1]\) and \(T(x) = \begin{cases} 
2x & \text{if } x \in [0, \frac{1}{2}] \\
-2x+2 & \text{if } x \in [\frac{1}{2}, 1]
\end{cases}\).

![Fig. 1. The tent map](image)

Clearly, any \(x\) close to 0 realizes the permutation 123 and any \(x\) close to 1 realizes the permutation 312. A simple case study shows that every \(x \in [0, 1/3]\) realizes the permutation \(\pi = 123\), every \(x \in [1/3, 2/5]\) realizes \(\pi = 132\), every \(x \in [2/5, 2/3]\) realizes \(\pi = 231\), every \(x \in [2/3, 4/5]\) realizes \(\pi = 213\), and finally, that every \(x \in [0, 1/3]\) realizes \(\pi = 312\). In particular, the permutation \(\pi = 321\) is not realizable.

The aim of this note is to provide a quick and understandable overview of the results of the following papers: [3], [8], [9], [4], [10] and [7]. Of course, I do not claim to be exhaustive; thus many interesting results will not be mentioned. I will end by listing two open questions in this field.

## 2 Permutations and full shifts

Let \(A_k\) denote the \(k\)-letter alphabet \(\{0, 1, \ldots, k - 1\}\) and consider the map \(\sigma_k : A_k^\infty \to A_k^\infty, (a_n) \mapsto (a_{n+1})\). This map is continuous with respect to the prefix metric on \(A_k^\infty\), for two distinct infinite words over \(A_k\), the longer is their common prefix, the closer they are. As the set \(A_k^\infty\) is compact with respect to this metric, \((A_k^\infty, \sigma_k)\) is a topological dynamical system. The domain \(A_k^\infty\) is usually called the full shift (over \(k\) symbols).
We use the notation $\tilde{\alpha}a_2\cdots a_i$ for the periodic sequence with period $a_1a_2\cdots a_i$, and $a_{i,\infty} = a_ia_{i+1}\cdots$ and $a_{i,j} = a_{i+1}\cdots a_{j-1}$. Moreover, for $(a_m)_{m\geq 1} \in A_k^N$, we let
\[ \tilde{\alpha} = \sup_{m\geq 1} a_{i,\infty}. \] (1)

In this section, we suppose that $A_k^N$ is ordered by the lexicographic order. We have
\[ \text{Pat}(\sigma_k, n, (a_m)_{m\geq 1}) = \pi \iff a_{[\pi^{-1}(1),\infty)} < \text{lex} a_{[\pi^{-1}(2),\infty)} < \text{lex} \cdots < \text{lex} a_{[\pi^{-1}(n),\infty)}. \]

Permutations in full shifts were first studied in [3]. In this paper, the authors show that the smallest permutations that are not allowed (such permutations are also said to be forbidden) in $(A_k^N, \sigma_k)$ have length $k + 2$. For example, for a binary alphabet, every permutation of length smaller than or equal to 3 is allowed, whereas it is easily checked that the permutation $\pi = 1423$ is not.

In [8], Elizalde is interested in computing the quantity $N_+(\pi)$, which is the smallest $k$ such that $\pi$ is realized in $(A_k^N, \sigma_k)$:
\[ N_+(\pi) = \min\{k \geq 1 : \pi \in A(\sigma_k)\}. \]

In Section 4, we will use the analogous notation $N_-(\pi)$ in the case of negative $\beta$-shifts. This is the reason why we write $N_+(\pi)$ instead of following Elizalde’s notation $N(\pi)$.

**Example 4.** Consider the permutation $\pi = 4217536 \in S_7$. Then any infinite sequence $(a_m)_{m\geq 1}$ starting with $21022120\cdots$ realizes $\pi$ since
\[
\begin{array}{cccccccccc}
2 & 1 & 0 & 2 & 2 & 1 & 2 & 0 & \cdots \\
4 & 2 & 1 & 7 & 5 & 3 & 6 \end{array}
\]
where, for each $m$, $1 \leq m \leq 7$, we wrote $\pi(m)$ below $a_m$ if $\pi(m) = i$. For instance, $a_{i,\infty} = 210\cdots < \text{lex} a_{[5,\infty)} = 212\cdots$, so $\pi(1) = 4 < \pi(5) = 5$. Note that we do not have uniqueness as $\text{Pat}(\sigma_3, 7, 21022120\cdots) = \text{Pat}(\sigma_3, 7, 21022122\cdots) = \pi$.

If $a_i a_{i+1} \cdots < \text{lex} a_j a_{j+1} \cdots$ and $a_i = a_j$ then $a_{i+1} a_{i+2} \cdots < \text{lex} a_{j+1} a_{j+2} \cdots$. If $a_i a_j \cdots$ realizes the permutation $\pi$, this means that $\pi(i) < \pi(j)$, then $a_i = a_j$ and $1 \leq i, j < n \implies \pi(i+1) < \pi(j+1)$. Thus, for a permutation $\pi \in S_n$, it is natural to consider the circular permutation
\[ \tilde{\pi} = (\pi(1) \pi(2) \cdots \pi(n)). \] (2)

Roughly, $N_+(\pi)$ is approximately equal to the number of descents in $\tilde{\pi}$, i.e., the number of indices $k < n$ such that $\tilde{\pi}(k) > \tilde{\pi}(k+1)$. Indeed, if $1 \leq i, j < n$, $\pi(i) < \pi(j)$, and $\pi(i+1) = \tilde{\pi}(\pi(i)) > \tilde{\pi}(\pi(j)) = \pi(j+1)$, then $a_i < a_j$. So, for each descent in $\tilde{\pi}$ where $\pi(1)$ is ignored we need one more symbol in order to realize $\pi$. 

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Example 5. We continue Example 4. One has \( \hat{\pi} = (4217536) = 17162345 \) and \( \hat{\pi} \) where \( \pi(1) = 4 \) is ignored, which is the sequence 716235, has 2 descents. By using the previous argument, we need at least 3 symbols to realize \( \pi \): 0, 1, 2. More precisely, the permutation \( \hat{\pi} \) also tells us the number of occurrences of those symbols in the prefix of length \( n-1 \) of any infinite sequence \((a_m)\) realizing the permutation \( \pi \):

\[
\hat{\pi} = 7162345 \\
0112222
\]

Then, the exact order of those \( n-1 \) digits in the prefix of any such \((a_m)\) is given by \( \pi \) itself:

\[
\pi = 4217536 \\
2102221
\]

The previous discussion ignores specific situations, where more symbols are needed. The main result of [8] is as follows:

**Theorem 6 ([8]).** Let \( n \geq 2 \). For any \( \pi \in S_n \),

\[
N_+(\pi) = 1 + \text{des}(\hat{\pi}) + \epsilon_+(\pi)
\]

where \( \text{des}(\hat{\pi}) \) is the number of descents in \( \hat{\pi} \) with \( \pi(1) \) removed and

\[
\epsilon_+(\pi) = \begin{cases} 
1 & \text{if } \pi \text{ ends with 21 or with } (n-1)n, \\
0 & \text{otherwise}.
\end{cases}
\]

Pursuing the previous discussion, in the case \( \epsilon_+(\pi) = 0 \) the prefix \( z_1z_2 \cdots z_{n-1} \) of any infinite sequence realizing the permutation \( \pi \) is given by

\[
z_j = \#\{1 \leq i < \pi(j) : \text{either } i \not\in \{\pi(n)-1, \pi(n)\} \text{ and } \hat{\pi}(i) > \hat{\pi}(i+1), \text{ or } i = \pi(n)-1 \text{ and } \hat{\pi}(i) > \hat{\pi}(i+2)\} \quad (3)
\]

where it should be understood that \( z_j \) is really the digit corresponding to this number.

Example 7. We continue Example 5. We have \( \epsilon_+(\pi) = 0 \) and by (3) we find \( z_1z_2 \cdots z_{n-1} = 2102221 \), as desired.

Example 8. Let \( \pi = 346752189 \). Then \( \hat{\pi} = (346752189) = 814627593 \) and \( \epsilon_+(\pi) = 1 \). In order to realize \( \pi \), an infinite word \( a_1a_2a_3 \cdots \) starting with \( z_1z_2 \cdots z_{n-1} = 11232103 \) needs one more symbol. Indeed

\[
a_n a_{n+1} \cdots \geq \text{lex} \ z_{n-1}a_n \cdots = 3a_n \cdots \implies a_n > 3
\]

and any infinite sequence starting with 112321034 realizes \( \pi \).
Example 9. Let $\pi = 24153$. Then $\hat{\pi} = (24153) = 54213$ and $\varepsilon_4(\pi) = 0$. Then $z_1 z_2 \cdots z_{n-1} = 1202$ (the prefix defined by (3)). Any sequence starting with 1202121 or 1202201 realizes $\pi$. This illustrates that, unlike the prefix of length $n-1$, the $n$th letter is not fixed by the permutation. This choice comes specifically from the descent 41 in $\hat{\pi}$ where $\pi(1) = 2$ is removed.

As a corollary of Theorem 6, Elizalde obtains that for $n \geq 3$ and $\pi \in S_n$, one has $N_+ (\pi) \leq n - 1$. In addition, he proves that for all $n \geq 3$, there are exactly 6 permutations $\pi \in S_n$ such that $N_+ (\pi) = n - 1$. These 6 permutations are:

\[1n2(n-1)3(n-2)\ldots, \ldots(n-2)3(n-1)2n1,\]
\[n1(n-1)2(n-2)3\ldots, \ldots3(n-2)1(n-1)1n,\]
\[\ldots4(n-1)3n21, \ldots(n-3)1(n-2)1(n-1)n.\]

In doing so, he answers a conjecture from [3]. In fact, Elizalde shows much more by proving a closed formula for the number $a_{n,N}$ of permutations $\pi$ of length $n$ for which $N_+(\pi) = N$, for any $n$ and $N$. In particular, for each fixed $N$, one has $a_{n,N} \sim n^{N^n-1}$ as $n$ tends to infinity, whence for each $k$, $\lim_{n \to \infty} \frac{1}{n} \log \#A(\sigma_k,N) = \lim_{n \to \infty} \frac{1}{n} \log (\sum_{N=1}^{k} a_{n,N}) = \log k$, in accordance with Theorem 1.

To end this section, let me also mention the work [4] where the authors consider other orderings of the elements of the full shift in the case of periodic orbits.

3 Permutations and positive $\beta$-shifts

Let $\beta > 1$. The $\beta$-transformation is the map $T_\beta : [0,1) \to [0,1), x \mapsto \{\beta x\}$ where $\{\cdot\}$ designates the fractional part of a real number. Instead of numbers $x \in [0,1)$, we will rather consider their $\beta$-expansions [15]:

\[x = \sum_{k=1}^{\infty} \frac{d_{\beta,k}(x)}{\beta^k} \text{ with } d_{\beta,k}(x) = \lfloor \beta T_{\beta}^{k-1}(x) \rfloor.\]

Set $d_{\beta}(x) = d_{\beta,1}(x) d_{\beta,2}(x) \cdots$. The $\beta$-shift is the topological closure of the set $\{d_{\beta}(x) : x \in [0,1)\}$ of all $\beta$-expansions from $[0,1)$; it is denoted by $\Omega_\beta$. Then $\sigma_\beta$ denotes the shift map $\sigma_\beta : \Omega_\beta \to \Omega_\beta$, $(a_n) \mapsto (a_{n+1})$. This map is continuous and the $\beta$-shift is a compact metric space, hence $(\Omega_\beta, \sigma_\beta)$ is a topological dynamical system. For all $x, y \in [0,1)$, we have $\sigma_\beta(d_{\beta}(x)) = d_{\beta}(T_{\beta}(x))$ and $x < y \iff d_{\beta}(x) <_{\text{lex}} d_{\beta}(y)$. Thus, for all $x \in [0,1)$ and all $n \geq 1$, we have

\[\text{Pat}(T_{\beta}, n, x) = \text{Pat}(\sigma_{\beta}, n, d_{\beta}(x)),\]

with the lexicographic order on $\Omega_\beta$. We note that if $a_1 a_2 \cdots = \lim_{n \to \infty} d_{\beta}(x_i)$ with $(x_i)$ a sequence of $[0,1)$, then for all sufficiently large $i$ and all $n \geq 1$, we have $\text{Pat}(\sigma_{\beta}, n, a_1 a_2 \cdots) = \text{Pat}(\sigma_{\beta}, n, d_{\beta}(x_i))$. Therefore $\mathcal{A}(T_{\beta}) = \mathcal{A}(\sigma_{\beta})$. Moreover,
if $1 < \beta < \beta'$, then $d_\beta(1) \leq d_{\beta'}(1)$, whence $\Omega_\beta \subseteq \Omega_{\beta'}$ and $A(T_\beta) \subseteq A(T_{\beta'})$ (this follows from Parry’s theorem, which characterizes the $\beta$-shift [14]).

In [9], Elizalde introduces the notion of the shift complexity of a permutation. We will take the liberty of calling it the positive shift complexity as we will need an analogous definition in the next section for negative $\beta$-shifts. The positive shift complexity of a permutation $\pi \in S_n$ is the quantity

$$B_+(\pi) = \inf\{\beta > 1 : \pi \in A(T_\beta)\}. \quad (4)$$

The main result of [9] is a method to compute $B_+(\pi)$. For $\pi \in S_n$, let $z_1 z_2 \cdots z_{n-1}$ as in (3). Moreover, let

$$m = \pi^{-1}(n) \quad \text{and} \quad \ell = \pi^{-1}(\pi(n) - 1) \text{ if } \pi(n) \neq 1. \quad (5)$$

For a sequence $a = a_1 a_2 \cdots$ of finitely many nonnegative digits such that $a = \bar{a}$ (see (1)), let $b_+(a)$ be the unique solution $\beta \geq 1$ of

$$\sum_{j=1}^{\infty} \frac{a_j}{\beta^j} = 1.$$

Note that when $a$ is an eventually periodic sequence, $b_+(a)$ is the unique real root greater than or equal to 1 of a polynomial.

**Theorem 10.** [9] Let $\pi \in S_n$. Then $\pi \in A(T_\beta) \iff \beta > b_+(\pi)$ where

$$a = \begin{cases} z_{\lfloor \pi(n) \rfloor - [\pi(n)-1]} & \text{if } \pi(n) \neq 1, \\ z_{\lfloor \pi(n) \rfloor} & \text{if } \pi(n) = 1 \text{ and } \pi(n-1) \neq 2, \\ z'_{\lfloor \pi(n) \rfloor - [\pi(n)-1]} & \text{if } \pi(n) = 1 \text{ and } \pi(n-1) = 2. \end{cases}$$

where the digits $z_j$ are defined as in (3) and for every $1 \leq j < n$, $z'_j = z_j + 1$. In particular, $B_+(\pi) = b_+(a)$ and $B_+(\pi)$ is 1 or a Parry number, i.e., a number $\beta > 1$ such that $d_\beta(1)$ is eventually periodic.

It directly follows from this theorem that $N_+(\pi) = 1 + \lfloor B_+(\pi) \rfloor$.

### 4 Permutations and negative $\beta$-shifts

In this section, I report recent results obtained by Steiner and myself [7]. Equivalent results were obtained simultaneously by Elizalde and Moore [10].

Let $\beta > 1$. Here we are interested in the $(-\beta)$-transformation $T_{-\beta}: (0, 1] \to (0, 1], x \mapsto [\beta x] + 1 - \beta x$. This maps is a generalization of $T_\beta$ in the following sense: $T_{-\beta}(x) = \{-\beta x\}$, except for the (finitely many) following values of $x$: $\frac{1}{\beta}, \frac{2}{\beta}, \ldots, \frac{\lfloor \beta \rfloor}{\beta}$.

Again, instead of numbers $x \in (0, 1]$, we will rather consider their $(-\beta)$-expansions [11, 16]:

$$x = -\sum_{k=1}^{\infty} \frac{d_{-\beta,k}(x) + 1}{(-\beta)^k} \text{ with } d_{-\beta,k}(x) = \lfloor \beta T_{-\beta}^{k-1}(x) \rfloor.$$
Set $d_{-\beta}(x) = d_{-\beta,1}(x)d_{-\beta,2}(x)\cdots$. For all $x, y \in (0, 1]$, we have $\sigma_{-\beta}(d_{-\beta}(x)) = d_{-\beta}(T_{-\beta}(x))$ and $x < y$ if and only if $d_{-\beta}(x) <_\text{alt} d_{-\beta}(y)$. Here we use the alternating lexicographic order for sequences:

$$a_1a_2\cdots <_\text{alt} b_1b_2\cdots \iff \exists i \geq 1, a_1\cdots a_{i-1} = b_1\cdots b_{i-1} \text{ and } \begin{cases} a_i < b_i & \text{if } i \text{ is odd}, \\ a_i < b_i & \text{if } i \text{ is even}. \end{cases}$$

The closure of the set of all $(-\beta)$-expansions $\{d_{-\beta}(x) : x \in (0, 1]\}$ forms the $(-\beta)$-shift, which is denoted by $\Omega_{-\beta}$. The shift map $\sigma_{-\beta} : \Omega_{-\beta} \to \Omega_{-\beta}$, $(a_m) \mapsto (a_{m+1})$ is continuous. For all $x \in (0, 1]$, one has

$$\text{Pat}(x, T_{-\beta}, n) = \text{Pat}(d_{-\beta}(x), \sigma_{-\beta}, n),$$

with the alternating lexicographic order on the $(-\beta)$-shift. Therefore $\mathcal{A}(T_{-\beta}) = \mathcal{A}(\sigma_{-\beta})$. From [16], we know that if $1 < \beta < \beta'$ then $d_{-\beta}(1) <_\text{alt} d_{-\beta'}(1)$ and $\Omega_{-\beta} \subseteq \Omega_{-\beta'}$, whence $\mathcal{A}(T_{-\beta}) \subseteq \mathcal{A}(T_{-\beta'})$.

Similarly to (4), the negative shift complexity of a permutation $\pi \in \mathcal{S}_n$ is the quantity

$$B_{-}(\pi) = \inf\{\beta > 1 : \pi \in \mathcal{A}(T_{-\beta})\}.$$

Let $\varphi$ be the substitution defined by $\varphi(0) = 1$, $\varphi(1) = 100$, with the unique fixed point $u = \varphi(u)$, i.e.,

$$u = 100111001001001110011\cdots.$$

If $\tilde{a} = a$ (as in (1) though we use the alternating lexicographic order in this case) and $a \leq u$, we set $b_{-}(a) = 1$. If $\tilde{a} = a$ and $a >_\text{alt} u$, then let $b_{-}(a)$ be the largest positive root of $1 + \sum_{j=1}^{\infty}(a_j + 1)(-x)^{-j}$ [10]. If $a$ is eventually periodic with preperiod of length $q$ and period of length $p$, then $b_{-}(a)$ is the largest positive solution of

$$(-x)^{p+q} + \sum_{k=1}^{p+q}(a_k + 1)(-x)^{p-q-k} = (-x)^q + \sum_{k=1}^{q}(a_k + 1)(-x)^{q-k}.$$

Since we are dealing with an order different from the lexicographic order, the discussion from Section 2 about the first $n-1$ digits of any sequence realizing a given permutation has to be adapted (see the examples at the end of this section). We define $n-1$ digits $z_1z_2\cdots z_{n-1}$ by

$$z_j = \# \{1 \leq i < \pi(j) : \text{either } i \notin \{\pi(n) - 1, \pi(n)\} \text{ and } \hat{\pi}(i) < \hat{\pi}(i + 1), \text{ or } i = \pi(n) - 1 \text{ and } \hat{\pi}(i) < \hat{\pi}(i + 2)\}$$

where it should be understood that $z_j$ is really the digit corresponding to this number. So, roughly, we now have one new digit for each ascent in $\hat{\pi}$ where $\pi(1)$ is removed (see Theorem 13 below). Let $m, \ell$ as in (5) and

$$r = \pi^{-1}(\pi(n) + 1) \text{ if } \pi(n) \neq n.$$
When
\[ z_{[r,n]} = z_{[r,n]}^{[r,n]} \quad \text{or} \quad z_{[r,n]} = z_{[r,n]}^{[r,n]}, \quad \text{if} \ \pi(n) \notin \{1, n\}, \quad (6) \]
we also use the following digits: for \( 0 \leq i < |r - \ell|, 1 \leq j < n, \)
\[ z_j^{(i)} = z_j + \begin{cases} 1 & \text{if} \ \pi(j) \geq \pi(r + i) \text{ and } i \text{ is even, or } \pi(j) \geq \pi(\ell + i) \text{ and } i \text{ is odd,} \\ 0 & \text{otherwise} \end{cases} \]
where, again, \( z_j^{(i)} \) really is the digit corresponding to this number.

**Theorem 11.** [7, 10] Let \( \pi \in S_n \) and \( \beta > 1. \) Then \( \pi \in A(T_{-\beta}) \iff \beta > b_{-\{a\}} \)
where
\[
a = \begin{cases} z_{[m,n]} z_{[r,n]} & \text{if } n - m \text{ is even, } \pi(n) \neq 1, \text{ and (6) does not hold,} \\ \min_{0 \leq i < |r - \ell|} z_{[m,n]}^{(i)} z_{[r,n]}^{-1} & \text{if } n - m \text{ is even, } \pi(n) \neq 1, \text{ and (6) holds,} \\ z_{[m,n]} z_{[r,n]} & \text{if } n - m \text{ is even and } \pi(n) = 1, \\ \min_{0 \leq i < |r - \ell|} z_{[m,n]}^{(i)} z_{[r,n]}^{-1} & \text{if } n - m \text{ is odd and (6) does not hold,} \\ 0 & \text{if } n - m \text{ is odd and (6) holds.} \end{cases} \quad (7)
\]
In particular \( B_{-\{\pi\}} = b_{-\{a\}} \) and if \( a >_{\text{alt}} u \), then \( B_{-\{\pi\}} \) is a Perron number, i.e., an algebraic integer all of whose Galois conjugates \( \alpha \) satisfying \( |\alpha| < b_{-\{a\}} \).

**Theorem 12.** [7] Let \( \pi \in S_n \) and \( a \) as in (7). We have \( B_{-\{\pi\}} = 1 \) if and only if \( a = \varphi^k(0) \) for some \( k \geq 0 \).

**Theorem 13.** [7, 10] Let \( \pi \in S_n \) and \( a \) as in (7). Then the minimal number of distinct symbols of a sequence \( w \) satisfying \( \text{Pat}(w, \sigma_{-\beta}, n) = \pi \) is
\[ N_{-\{\pi\}} = 1 + \lceil B_{-\{\pi\}} \rceil = 1 + \text{asc}(\hat{\pi}) + \epsilon_{-\{\pi\}}, \]
where \( \text{asc}(\hat{\pi}) \) denotes the number of ascents in \( \hat{\pi} \) with \( \hat{\pi}(\pi(n)) = \pi(1) \) removed and
\[ \epsilon_{-\{\pi\}} = \begin{cases} 1 & \text{if (6) holds or } a = \text{asc}(\hat{\pi})0, \\ 0 & \text{otherwise.} \end{cases} \]
In particular, we have \( N_{-\{\pi\}} \leq n - 1 \) for all \( \pi \in S_n, \) \( n \geq 3, \) with equality for \( n \geq 4 \) if and only if
\[ \pi \in \{12 \cdots n, \ 12 \cdots (n-2) n(n-1), \ n(n-1) \cdots 1, \ n(n-1) \cdots 312\}. \]

**Example 14.**
1. Let \( \pi = 3421. \) Then \( n = 4, \hat{\pi} = 3142, \ z_{[1,4]} = 110, \ m = 2, \pi(n) = 1, \ r = 3. \)
We obtain that \( a = \varphi^{2}(0) = 100 = \varphi^{2}(0), \) thus \( B_{-\{\pi\}} = b_{-\{a\}} = 1. \) Indeed, we have \( \text{Pat}(110010011, \sigma_{-\beta}, n) = \pi. \)
2. Let $\pi = 892364157$. Then $n = 9$, $\hat{\pi} = 536174892$, $z_{(1,9)} = 33012102$, $m = 2$, $\ell = 5$, $r = 1$, thus $a = z_{(2,9)} = 301210223$, and $b_-(a)$ is the unique root $x > 1$ of

$$x^8 - 4x^7 + x^6 - 2x^5 + 3x^4 - 2x^3 + x^2 - 3x + 1 = 0.$$ 

We get $B_-(\pi) \approx 3.831$, and we have $\text{Pat}(330121023, 301210223, \sigma_{-\beta}, n) = \pi$.

3. Let $\pi = 453261$. Then $n = 6$, $\hat{\pi} = 462531$, $z_{(1,6)} = 11001$, $m = 5$, $\pi(n) = 1$, $r = 4$, thus $a = z_{5, 623} = 110$, and $b_-(a) = 2$. We have $\text{Pat}(110010, 2, \sigma_{-\beta}, n) = \pi$.

4. Let $\pi = 7325416$. Then $n = 7$, $\hat{\pi} = 6521473$, $z_{(1,7)} = 100100$, $m = r = 1$, $\ell = 4$. Hence (6) holds, and $z_{(1,7)}^{(0)} = 200100$, $z_{(1,7)}^{(1)} = 200210$, $z_{(1,7)}^{(2)} = 211210$.

Since $n - m$ is even, we have

$$a = \min_{i \in \{0, 1, 2\}} \frac{z_{(1,7)}^{(i)}}{z_{(1,7)}} = \min\{200100, 200210, 211210\} = 211210.$$ 

Therefore, $B_-(\pi) \approx 2.343$ is the largest positive root of

$$0 = (x^6 - 3x^5 + 2x^4 - 2x^3 + 3x^2 - 2x + 1) - (-x^3 + 3x^2 - 2x + 2) = x^6 - 3x^5 + 2x^4 - x^3 - 1.$$ 

We have $\text{Pat}(211210)^{2k+2}, 2, \sigma_{-\beta}, n) = \pi$ for $k \geq 0$.

5 Comparing the positive and negative $\beta$-shifts

In Table 1, we give the values of the shift complexity $B(\pi)$ for all permutations of length up to 4, and we compare them with the values obtained by [9] for the positive $\beta$-shift. Here $B(\pi)$ has to be understood as $B_-(\pi)$ or $B_+(\pi)$ accordingly. Note that much more permutations satisfy $B_-(\pi) = 1$ for the negative $\beta$-shift than $B_+(\pi) = 1$ for the positive one.

6 Open problems

Let me conclude with two open problems.

- Count all permutations with $B_-(\pi) \leq N$ or $B_-(\pi) < N$, in particular with $B_-(\pi) = 1$. From Theorem 1 we know that

$$\lim_{n \to \infty} \frac{1}{n} \log \#\{\pi \in S_n : B_-(\pi) < \beta\} = \lim_{n \to \infty} \frac{1}{n} \log \#\{\pi \in S_n : B_-(\pi) \leq \beta\} = \log \beta$$

What are the precise asymptotics of

$$c_n = \#\{\pi \in S_n : B_-(\pi) = 1\}?$$

The first values are given by $(c_n)_{2 \leq n \leq 9} = 2, 5, 12, 19, 34, 57, 82, 115$.

- Describe the permutations given by the transformations

$$T_{\beta, \alpha} : [0, 1) \to [0, 1), \ x \mapsto \beta x + \alpha \mod \beta x + \alpha.$$
Table 1. \( B(\pi) \) for the \((-\beta)\)-shift and the \(\beta\)-shift, for all permutations of length up to 4.

<table>
<thead>
<tr>
<th>( B(\pi) )</th>
<th>( \pi ), negative (\beta)-shift</th>
<th>( \pi ), positive (\beta)-shift</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \beta - 1 )</td>
<td>12, 21</td>
</tr>
<tr>
<td></td>
<td></td>
<td>123, 132, 213, 321, 312</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1324, 1342, 1432, 2134, 2143, 2314, 2431, 3142, 3214, 3421, 4213</td>
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<tr>
<td>1.465</td>
<td>( \beta^4 - \beta^3 - 1 )</td>
<td>1342, 2413, 3124, 4231</td>
</tr>
<tr>
<td>1.618</td>
<td>( \beta^2 - \beta - 1 )</td>
<td>132, 213, 321</td>
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<td></td>
<td></td>
<td>1423, 3412, 4231</td>
</tr>
<tr>
<td>1.755</td>
<td>( \beta^2 - 2\beta^3 + \beta - 1 )</td>
<td>1243, 1324, 2431, 3142, 4312</td>
</tr>
<tr>
<td>1.802</td>
<td>( \beta^2 - 2\beta^2 - 2\beta + 1 )</td>
<td>4213</td>
</tr>
<tr>
<td>1.839</td>
<td>( \beta^2 - \beta^2 - \beta - 1 )</td>
<td>4132</td>
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<tr>
<td></td>
<td>( \beta - 2 )</td>
<td>1432, 2143, 3214, 4321</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2134, 3241</td>
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<tr>
<td>2.247</td>
<td>( \beta^2 - 2\beta^3 + \beta + 1 )</td>
<td>4321</td>
</tr>
<tr>
<td>2.414</td>
<td>( \beta^2 - 2\beta - 1 )</td>
<td>2314, 4321</td>
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<td>2.618</td>
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</tr>
<tr>
<td>2.732</td>
<td>( \beta^2 - 2\beta - 2 )</td>
<td>4312</td>
</tr>
</tbody>
</table>

7 Acknowledgements

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References

7. Charlier, É., Steiner, W.: Permutations and negative beta-shifts, working paper